A numerical approach for solving a class of nonlinear functional differential equations

Burcu Gürbüz

Institute of Mathematics, Johannes Gutenberg-University Mainz, Staudingerweg 9, 55128 Mainz, Germany
E-mail: burcu.gurbuz@uni-mainz.de

Abstract

In this study, a special class of nonlinear functional differential equations with mixed conditions has been investigated with regard to Laguerre polynomials to find numerical solutions of these problems. Those mathematical dynamics are of growing important in the natural sciences, in engineering, in economics, in medical and life sciences, and in emerging space-time design and research. Our novel numerical approach has been applied on the related problem and its efficiency has been proved by error analysis on an illustrative example. The high efficiency of our now method is document by a numerical example. The paper ends with a conclusion, scientific discussions about structural frontiers, and an outlook to future investigations and emerging applications.

2000 Mathematics Subject Classification. **65L03**. 33C45 65B99.
Keywords. Functional differential equations, Laguerre polynomials, convergence.

1 Introduction

Nonlinear functional differential equations are of an importance in applied sciences such as in the area of biology, chemistry, physics and engineering, but also medicine, environmental, space-time, neuro- and cognitive and information sciences. These types of differential equations provide a wide range of applications in the areas which explain more complicated scenarios due to the nonlinear functional terms in the equations [1], [2], [3]. Even though the general theory of the nonlinear functional differential equations and fundamental results has been investigated, the numerical aspect for the solution of the problems together with the mixed conditions still under investigation. The problems of applied sciences are interdisciplinary and tools are developed with of an adaptable concept [4]. Thus, in particular, such models are investigated with the help of numerical approaches. Developing methods and improving approximation techniques for problem solution lead numerically efficient results with the help of these modifications [5]. Besides, numerical methods help for obtaining more understanding of specifical mathematical models as well as their behaviours. Due to this investigation a mathematical model for a physical or biological dynamics is leading a strong way of understanding a real phenomena [6].

The numerical solutions of these problems which address these types of equations subject to the mixed conditions have been studied by many authors. In particular, Tau method was introduced by Ortiz [7], Lagrange interpolation and Chebyshev interpolation were presented by Rashed [8], Spectral methods were given by Venturi et al. [9], stepsize Runge-Kutta method was described by Wang et al. [10], and so on. Besides, block methods, Splines and Hermte methods, multistep methods, Chebyshev series and the tau method, nonsmooth solutions, and the details of these methods with

Tbilisi Mathematical Journal Special Issue (8 - 2021), pp. 1–11.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 03 June 2020.
Accepted for publication: 02 November 2020.

the applications on nonlinear functional differential equations were explained in the book of Cryer [11]. Such methods are very beneficial for finding the solutions of nonlinear differential equations which are also advantageous for giving understanding the models including these types of equations. So, the numerical methods are also rewarding in order to build networks between different branches of science (cf. [12]-[15]).

The organization of the manuscript as follows. In Section 2 mathematical model is introduced with its details and together with the mixed conditions. In Section 3 the numerical method is explained clearly. The convergence and accuracy of the method is presented in Section 4. In Section 5, a numerical example is interpreted and discussed. Conclusion remarks, discussions about obstacles of various kinds and how to overcome them by future research and design and discussion of the results are summarized in Section 6. Consequently, an acknowledgment of the study is given in Section 7 [16].

2 Mathematical model

In this section, the mathematical model is explained. In this work, we study on the numerical solutions of a class of nonlinear functional differential equations which include second order nonlinear terms in the form

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(\alpha_k x+\beta_k(x))+\sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x)y^{(p)}(x)y^{(q)}(x)=g(x),\ x\in[a,b], \tag{2.1}$$

with the mixed conditions

$$\sum_{k=0}^{m-1}(a_{kj}y^{(k)}(a)+b_{kj}y^{(k)}(b))=\lambda_j,\ j=0,1,\ldots,m-1, \tag{2.2}$$

within $P_k(x)$, $Q_{pq}(x)$, and $g(x)$ are continuous functions in the interval $0\leq a\leq x\leq b<\infty$ (see [1]-[3]). Our goal is to acquire the approximate solution in truncated Laguerre series form:

$$y(x)\cong y_N(x)=\sum_{n=0}^{N} a_n L_n(x), \quad a\leq x\leq b, \tag{2.3}$$

where a_n, $n=0,1,\ldots,N$, are unknown coefficients, and the terms $L_n(x)$ denote the Laguerre polynomials [17], [18].

3 Numerical method

In this section, the numerical method is introduced. Primarily, the matrix relation of Eq. (2.3) is described as

$$[y(x)]\cong y_N(x)=\mathbf{L}(x)\mathbf{A}=\mathbf{X}(x)\mathbf{H}\mathbf{A}, \tag{3.1}$$

where

$$\mathbf{L}(x) = \begin{bmatrix} L_0(x) & L_1(x) & \dots & L_N(x) \end{bmatrix},$$
$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & \dots & x^N \end{bmatrix},$$
$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \dots & a_N \end{bmatrix} \text{ and}$$
$$\mathbf{H} = \begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & \frac{(-1)^0}{0!}\binom{1}{0} & \cdots & \frac{(-1)^0}{0!}\binom{N}{0} \\ 0 & \frac{(-1)^1}{1!}\binom{1}{1} & \cdots & \frac{(-1)^1}{1!}\binom{N}{1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}. \tag{3.2}$$

Besides, the matrix form of $y^{(k)}(x)$ and together with the term in Eq. (2.1) include derivative is defined also from Eq. (3.1) (cf. [17]-[19]) as

$$[y^{(k)}(t)] = \mathbf{X}(t)\mathbf{H}\mathbf{B}\mathbf{A}_i, \tag{3.3}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.4}$$

We also describe the matrix form of $y(\alpha_k x + \beta_k(x))$ in Eq. (2.1) from Eqs. (3.1) and (3.2), respectively, as

$$[y(\alpha_k x + \beta_k(x))] = \mathbf{X}(\alpha_k x + \beta_k(x))\mathbf{H}\mathbf{A} = \mathbf{X}(x)\mathbf{M}(\alpha_k, \beta_k(x))\mathbf{H}\mathbf{A}, \tag{3.5}$$

and

$$[y^{(k)}(\alpha_k x + \beta_k(x))] = \mathbf{X}(\alpha_k x + \beta_k(x))\mathbf{B}^{(k)}\mathbf{H}\mathbf{A} = \mathbf{X}(x)\mathbf{M}(\alpha_k, \beta_k(x))\mathbf{B}^{(k)}\mathbf{H}\mathbf{A}, \tag{3.6}$$

where

$$\mathbf{M}(\alpha_k, \beta_k(x)) = \begin{bmatrix} \binom{0}{0}(\alpha_k)^0(\beta_k)^0 & \binom{1}{0}(\alpha_k)^0(\beta_k)^1 & \cdots & \binom{N}{0}(\alpha_k)^0(\beta_k)^N \\ 0 & \binom{1}{1}(\alpha_k)^1(\beta_k)^0 & \cdots & \binom{N}{1}(\alpha_k)^1(\beta_k)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(\alpha_k)^N(\beta_k)^0 \end{bmatrix}.$$

Moreover, the nonlinear terms of Eq. (2.1) is written in the matrix form by means of Eqs. (3.1)-(3.2) (cf. [20]-[22]):

$$\begin{aligned}
[y^{(0)}(x)]^2 &= \mathbf{X}(x)\mathbf{H}\bar{\mathbf{X}}(x)\bar{\mathbf{H}}\bar{\mathbf{A}}, \\
[y^{(1)}(x)][y^{(0)}(x)] &= \mathbf{X}(x)\mathbf{H}\mathbf{B}\bar{\mathbf{X}}(x)\bar{\mathbf{H}}\bar{\mathbf{A}}, \\
[y^{(1)}(x)]^2 &= \mathbf{X}(x)\mathbf{H}\mathbf{B}\bar{\mathbf{X}}(x)\bar{\mathbf{H}}\bar{\mathbf{B}}\bar{\mathbf{A}}, \\
[y^{(2)}(x)][y^{(1)}(x)] &= \mathbf{X}(x)\mathbf{H}\mathbf{B}^2\bar{\mathbf{X}}(x)\bar{\mathbf{H}}\bar{\mathbf{B}}\bar{\mathbf{A}}, \\
[y^{(2)}(x)][y^{(0)}(x)] &= \mathbf{X}(x)\mathbf{H}\mathbf{B}^2\bar{\mathbf{X}}(x)\bar{\mathbf{H}}\bar{\mathbf{A}}, \\
[y^{(2)}(x)]^2 &= \mathbf{X}(x)\mathbf{H}\mathbf{B}^2\bar{\mathbf{X}}(x)\bar{\mathbf{H}}\bar{\mathbf{B}}^2\bar{\mathbf{A}},
\end{aligned}$$

where

$$\begin{aligned}
\bar{\mathbf{H}} &= \operatorname{diag}\left[\mathbf{H}, \mathbf{H}, ..., \mathbf{H}\right], \\
\bar{\mathbf{B}} &= \operatorname{diag}\left[\mathbf{B}, \mathbf{B}, ..., \mathbf{B}\right], \\
\bar{\mathbf{X}} &= \operatorname{diag}\left[\mathbf{X}, \mathbf{X}, ..., \mathbf{X}\right], \\
\bar{\mathbf{B}}^2 &= \operatorname{diag}\left[\mathbf{B}^2, \mathbf{B}^2, ..., \mathbf{B}^2\right], \\
\bar{\mathbf{A}}^2 &= \operatorname{diag}\left[a_0\mathbf{A}, a_1\mathbf{A}, ..., a_N\mathbf{A}\right].
\end{aligned}$$

Definition 3.1. The *collocation points* are defined by

$$x_s = a + \frac{b-a}{N}s,\ s = 0, 1, \ldots, N, \tag{3.7}$$

and $a \leq x_0 < x_1 < \ldots < x_N \leq b$. Then we replace the collocation points of Eqs. (3.7) in Eq. (3.3) [23]:

$$\sum_{k=0}^{m} P_k(x_s)y^{(k)}(\alpha_k x_s + \beta_k(x_s)) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_s)y^{(p)}(x_s)y^{(q)}(x_s) = g(x_s);$$

$$\sum_{k=0}^{m} \mathbf{P}_k(x_s)\mathbf{X}(x_s)\mathbf{M}(\alpha_k, \beta_k(x_s))\mathbf{B}^k\mathbf{H}\mathbf{A} + \sum_{p=0}^{2}\sum_{q=0}^{p} \mathbf{Q}_{pq}(x_s)\mathbf{Y}^{(p,q)}\bar{\mathbf{A}} = \mathbf{G},$$

$$\sum_{k=0}^{m} \mathbf{P}_k\bar{\mathbf{X}}\bar{\mathbf{M}}_{\alpha_k,\beta_k}\bar{\mathbf{B}}^k\mathbf{H}\mathbf{A} + \sum_{p=0}^{2}\sum_{q=0}^{p} \mathbf{Q}_{pq}\mathbf{Y}^{(p,q)}\bar{\mathbf{A}} = \mathbf{G}, \tag{3.8}$$

where

$$
\begin{aligned}
\bar{\mathbf{H}}^{*} &= \begin{bmatrix} \mathbf{H} \\ \mathbf{H} \\ \vdots \\ \mathbf{H} \\ \mathbf{H} \end{bmatrix}, \quad \bar{\mathbf{M}}(\alpha_k, \beta_k(x)) = \operatorname{diag}\left[\mathbf{M}(\alpha_k, \beta_k(x_0)), \mathbf{M}(\alpha_k, \beta_k(x_1)), ..., \mathbf{M}(\alpha_k, \beta_k(x_N))\right], \\
\mathbf{Y}^{(0,0)} &= \mathbf{XH\bar{X}\bar{H}\bar{A}}, \quad \mathbf{Y}^{(1,0)} = \mathbf{XHB\bar{X}\bar{H}\bar{A}}, \\
\mathbf{Y}^{(1,1)} &= \mathbf{XH\bar{X}\bar{H}\bar{B}\bar{A}}, \quad \mathbf{Y}^{(2,1)} = \mathbf{XHB^2\bar{X}\bar{H}\bar{B}\bar{A}}, \\
\mathbf{Y}^{(2,0)} &= \mathbf{XHB^2\bar{X}\bar{H}\bar{A}}, \quad \mathbf{Y}^{(2,2)} = \mathbf{XHB^2\bar{X}\bar{H}\bar{B}^2\bar{A}}.
\end{aligned}
$$

By identifying

$$\sum_{k=0}^{m} \mathbf{P}_k \bar{\mathbf{X}} \bar{\mathbf{M}}_{\alpha_k,\beta_k} \bar{\mathbf{B}}^k \mathbf{H}^* = \mathbf{W}$$

and

$$\sum_{p=0}^{2} \sum_{q=0}^{p} \mathbf{Q}_{pq} \mathbf{Y}^{(p,q)} = \mathbf{V}$$

in Eq. (3.8), we have the augmented matrix form:

$$\mathbf{WA} + \mathbf{V}\bar{\mathbf{A}} = \mathbf{G}, \quad [\mathbf{W}; \mathbf{V} : \mathbf{G}]. \tag{3.9}$$

Similarly, we apply the same procedure to the conditions which are defined in Eq. (2.2) and we get

$$\sum_{k=0}^{m-1} (a_{kj} \mathbf{X}(a) \mathbf{B}^k \mathbf{H} + b_{kj} \mathbf{X}(b) \mathbf{B}^k \mathbf{H}) \mathbf{A} = [\lambda_j], \tag{3.10}$$

or

$$\mathbf{UA} + \mathbf{0}^* \bar{\mathbf{A}} = \lambda.$$

In here, we have

$$
\begin{aligned}
\mathbf{U} &= \begin{bmatrix} u_{j0} & u_{j1} & \dots & u_{jN} \end{bmatrix} \; j = 0, 1, 2, ..., m-1, \\
\lambda &= \begin{bmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_N \end{bmatrix}^T \text{ and} \\
\mathbf{0}^* &= \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}^T.
\end{aligned}
$$

In order to have unknown matrices $\mathbf{A}$ and $\bar{\mathbf{A}}$, we replace the last rows of the augmented matrix of Eq. (3.9) by Eq. (3.10) and we get the required augmented matrix system as

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{V}} : \tilde{\mathbf{G}}]. \tag{3.11}$$

By using Gaussian Elimination procedure, we have the solution of the system in Eq. (3.11) and we find the Laguerre series solution by Eq. (2.3) (cf. [24]-[25]).

4 Convergence and error analysis

In this section, the convergence of the method and accuracy estimation are described [26], [27]. First we consider that the problem in (2.1)-(2.3) has a solution then its approximation $y_N(x)$, exists for every N which is sufficiently large enough and it converges with its derivatives to the solution $y(x)$. The results on the solutions and their convergence have been already considered (see [3], [8], [10], [28], [29]).

Here, the accuracy is checked for the obtained solutions by resulting equation.

Definition 4.1. The function $y_N(x)$ and the derivatives of the function and $x = x_j \in [0, T]\, j = 1, 2, \ldots$ are substituted in Eq. (2.1).

$$R_N(x_j) = \sum_{k=0}^{m} P_k(x_j) y^{(k)}(\alpha_k x_j + \beta_k(x_j)) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_j) y^{(p)}(x_j) y^{(q)}(x_j) - g(x_j) \cong 0. \quad (4.1)$$

is defined as *residual error function*. We also describe it as $R_N(x_j) \leq 10^{-r_j}$ for any r_j positive number (see [30]-[31]).

Definition 4.2. Moreover, we define the *absolute error function* as

$$E_N(x_j) = |y_N(x_j) - \sum_{k=0}^{m} P_k(x_j) y^{(k)}(\alpha_k x_j + \beta_k(x_j)) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_j) y^{(p)}(x_j) y^{(q)}(x_j) - g(x_j)| \cong 0.$$

Theorem 4.1. (Upper Bound Error) Suppose that $a \leq x \leq b$ and $N \in \mathbb{N}$. Then we have the upper bound error R_N as

$$\left| \int_a^b R_N(x)dx \right| \leq \int_a^b |R_N(x)|\, dx, \quad a \leq x \leq b, \quad N \in \mathbb{N}.$$

Proof. By using the Mean Value Theorem

$$\int_a^b R_N(x)dx = (b-a)R_N(c),\ a < c < b,$$

$$\left| \int_a^b R_N(x)dx \right| = (b-a)\, |R_N(c)|\, ,$$

so that

$$(b-a)\, |R_N(c)| \leq \int_a^b |R_N(x)|\, dx,$$

$$|R_N(x)| \leq \frac{\int_a^b |R_N(x)|\, dx}{(b-a)} = R_N.$$

Q.E.D.

5 Illustrative example

Here, we consider functional differential equation involving second-order nonlinear terms in a form which includes the conditions as

$$y^{''}(x) + \exp(x^2)y(x - x^2) + \exp(x)y^2(x) = \exp(3x),\ x \in [0, 1], \tag{5.1}$$

$$y(0) = 1 \quad \text{and} \quad y^{'}(0) = 1,\ x \in [0, 1]. \tag{5.2}$$

This functional differential equation with initial values has an exact solution as $y = \exp(x)$.

Comparison of the error functions are shown in Table 1. Besides, Figure 1 shows error functions with specific collocation point and also for the different N values. In Table 2., the upper bound error $\tilde{R}_N$ for $N = 3,\ 6,\ 8$ is shown.

TABLE 1. Absolute error comparison.

x_j	$E_3(x_j)$	$E_6(x_j)$	$E_8(x_j)$
0.0	0.563215E-5	0.689033E-6	0.365915E-7
0.5	0.513693E-4	0.725690E-5	0.650241E-6
1.0	0.630694E-4	0.945630E-5	0.678063E-6

TABLE 2. Upper bound error comparison.

$\tilde{R}_3$	$\tilde{R}_6$	$\tilde{R}_8$
0.13215E-4	0.11533E-5	0.16659E-6

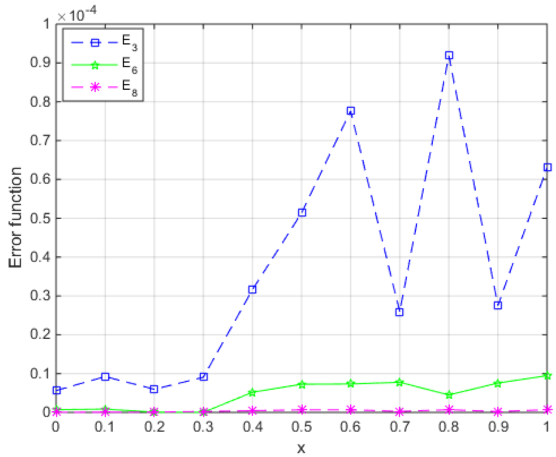

FIGURE 1. Error comparison for $N = 3, 6,\ 8$ of the test case.

6 Conclusion and outlook

The present numerical method has been applied on the problem Eqs. (2.1)-(2.2) with N different truncation limits where we see that the more efficient results are obtained for large enough N values. Besides, the straightforward steps of the method provide accurate results in a short period of time. The numerical technique therefore is applicable and recommended for the further applications in different types of mathematical models (see [32], [33], [34], [35]). Formerly, it was applied in dynamical systems (see [36], [37], [38]) and also still under preparation for the future adaptations (cf. [39], [40]). On the other hand, emerging and future applications will address information and rumor propagation (cf. [41]), on data science and artificial intelligence (cf. [42]), sustainable development (cf. [43]) as well as cognitive and neurosciences (cf. [44], [45], [46]). These human related, mental and generalize space-time studies are in a high need of a careful modeling, parametric and solution of dynamics. Mathematical and especially structural obstacles to be overcome in future research will be due our intended turn to stochastic functional and infinite-dimensional equations, and by the inclusion of non-Euclidean, curved geometries. These extensions are typical for generalized space-time design and research. Consequently, the technique is applicable with the help of some modifications in order to be used in different research fields.

7 Acknowledgement

The author thanks the Embassy of France (Turkey) for their financial support titled "Young Visiting Research Fellowship, 2019" and University of Nantes, Jean Leray Mathematics Laboratory, to use all the facilities required for part of the work. The author wishes to express her sincere gratitude to Gerhard-Wilhelm Weber (Poznan University of Technology), co-editors and the reviewers for constructive and competent criticism to have an improved version of the article.

References

[1] C. Corduneanu, Y. Li and M. Mahdavi, *Functional Differential Equations: Advances and Applications*, John Wiley & Sons (2016).

[2] N. V. Azbelev and L. Rakhmatullina, *Introduction to the theory of functional differential equations: methods and applications*, Hindawi Publishing Corporation **3** (2007).

[3] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press (2004).

[4] K. Schmitt, *Delay and functional differential equations and their applications*, Elsevier (2014).

[5] M. P. Fourman, *Applications of sheaves*, Springer-Verlag (1979).

[6] V. Lakshmikantham, *World Congress of Nonlinear Analysts'92: Proceedings of the First World Congress of Nonlinear Analysts, Tampa, Florida, August 19-26*, Walter de Gruyter (1996).

[7] E. L. Ortiz, *On the numerical solution of nonlinear and functional differential equations with the Tau method*, Springer, Berlin, Heidelberg (1978).

[8] M. T. Rashed, *Numerical solution of functional differential, integral and integro-differential equations*, Applied Mathematics and Computation **156** (2) (2016) 485–492.

[9] D. Venturi and A. Dektor, *Spectral methods for nonlinear functionals and functional differential equations*, arXiv preprint arXiv:2003.14308 (2020).

[10] W. Wang and C. Zhang, *Dissipativity of variable-stepsize Runge-Kutta methods for nonlinear functional differential equations with application to Nicholson's blowflies models*, Communications in Nonlinear Science and Numerical Simulation **97** (2021) 105723.

[11] C. W. Cryer, *"Numerical methods for functional differential equations." Delay and functional differential equations and their applications*, Academic Press (1972) 17–101.

[12] I. Dinur and K. Nissim, *Revealing information while preserving privacy*, In Proceedings of the twenty-second ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems **12**(2) (2003) 202–210.

[13] D. Sahoo, S. Mondal and G. P. Samanta, *Interaction among toxic phytoplankton with viral infection and zooplankton in presence of multiple time delays*, International Journal of Dynamics and Control (2020) 1–26.

[14] R. Abazari and A. Kılıçman, *Application of differential transform method on nonlinear integro-differential equations with proportional delay*, Neural Computing and Applications **24**(2) (2014) 391–397.

[15] F. H. Khedr and S. A. Abd El-Baki, *Fuzzy s-continuous functions and semilocally connected fuzzy topological spaces*, Fuzzy sets and systems **70**(1) (1995) 113–117.

[16] J. P. Chávez, B. Gürbüz, and C. M. Pinto, *The effect of aggressive chemotherapy in a model for HIV/AIDS-cancer dynamics*, Communications in Nonlinear Science and Numerical Simulation **75** (2019) 109–120.

[17] B. Gürbüz, M. Sezer and C. Güler, *Laguerre collocation method for solving Fredholm integro-differential equations with functional arguments*, Journal of Applied Mathematics **2014** (2014) 1–12.

[18] B. Gürbüz and M. Sezer, *Laguerre collocation method for solving Fredholm integro-differential equations with functional arguments*, Gazi University Journal of Science **29**(2) (2016) 419–434.

[19] D. Varol Bayram and A. Daşçıoğlu, *A method for fractional Volterra integro-differential equations by Laguerre polynomials*, Advances in Difference Equations **2018**(1) (2018) 466.

[20] B. Gürbüz and M. Sezer, *A Modified Laguerre Matrix Approach for Burgers–Fisher Type Nonlinear Equations*, Numerical Solutions of Realistic Nonlinear Phenomena Springer, Cham. (2020) 107–123.

[21] B. Gürbüz, *Laguerre matrix-collocation technique to solve systems of functional differential equations with variable delays*, AIP Conference Proceedings **2183**(1) (2019) 090007.

[22] B. Gürbüz and M. Sezer, *Laguerre polynomial solutions of a class of initial and boundary value problems arising in science and engineering fields*, Acta Physica Polonica A **130**(2016) 1194–1197.

[23] S. Y. Reutskiy, *The backward substitution method for multipoint problems with linear Volterra-Fredholm integro-differential equations of the neutral type*, Journal of Computational and Applied Mathematics **296** (2016) 724–738.

[24] B. Gürbüz, *A numerical scheme for the solution of neutral integro-differential equations including variable delay*, Mathematical Sciences (2021) https://doi.org/10.1007/s40096-021-00388-3.

[25] M. Özel and M. Sezer, *A numerical approach for a nonhomogeneous differential equation with variable delays*, Mathematical Sciences **12**(2) (2018) 145–155.

[26] A. Andò and D. Breda, *Convergence analysis of collocation methods for computing periodic solutions of retarded functional differential equations*, SIAM Journal on Numerical Analysis **58**(5) (2020) 3010–3039.

[27] B. A. J. de Wolff, F. Scarabel, S. M. Verduyn Lunel and O. Diekmann, *Pseudospectral approximation of Hopf bifurcation for delay differential equations*, SIAM Journal on Applied Dynamical Systems **20**(1) (2021) 333–370.

[28] A. Bellen, *The collocation method for the numerical approximation of the periodic solutions of functional differential equations*, Computing **23**(1) (1979) 55–66.

[29] H. T. Banks, *Approximation of nonlinear functional differential equation control systems*, Journal of Optimization Theory and Applications **29**(3) (1979) 383–408.

[30] F. A. Oliveira, *Collocation and residual correction*, Numerical Mathematics (1980) 27–31.

[31] K. Erdem Biçer and M. Sezer, *A computational method for solving differential equations with quadratic non-linearity by using Bernoulli polynomials*, Thermal Sciences **23**(1) (2019) 275–283.

[32] K. Maleknejad, S. Sohrabi and H. Derili, *A new computational method for solution of non-linear Volterra–Fredholm integro-differential equations*, International Journal of Computer Mathematics **87**(2) (2010) 327–338.

[33] K. Yıldırım, B. Ibis and M. Bayram, *New solutions of the nonlinear Fisher type equations by the reduced differential transform*, Nonlinear Sci. Lett. A **87**(2) (2012) 29–36.

[34] R. Abazari and K. Yıldırım, *Numerical study of Sivashinsky equation using a splitting scheme based on CrankNicolson method*, Mathematical Methods in the Applied Sciences **42**(16) (2019) 5509–5521.

[35] M. Bayram and K. Yıldırım, *Approximates Method for Solving an Elasticity Problem of Settled of the Elastic Ground with Variable Coefficients*, Natural Sciences Publishing (NSP).

[36] B. Gürbüz, *Laguerre matrix-collocation technique to solve systems of functional differential equations with variable delays*, AIP Conference Proceedings **2183**(1) (2019) 090007.

[37] B. Gürbüz, H. Mawengkang, I. Husein and G. W. Weber, *Rumour propagation: an operational research approach by computational and information theory*, Central European Journal of Operations Research **19**(1) (2021) 1–17.

[38] B. Gürbüz and M. Sezer, *Laguerre Matrix-Collocation Method to Solve Systems of Pantograph Type Delay Differential Equations*, International Conference on Computational Mathematics and Engineering Sciences, Springer, Cham. (2019) 121–132.

[39] M. Kara, D. T. Tollu and Y. Yazlık, *Global behavior of two-dimensional difference equations system with two periodic coefficients*, Tbilisi Mathematical Journal **42**(16) (2020) 49–64.

[40] S. Alkan and V. F. Hatipoğlu, *Approximate solutions of Volterra-Fredholm integro-differential equations of fractional order*, Tbilisi Mathematical Journal **10**(2) (2017) 1–13.

[41] S. Belen, E. Kropat and G. W. Weber, *On the classical MakiThompson rumour model in continuous time*, Central European Journal of Operations Research **19**(1) (2011) 1–17.

[42] P. Vasant, I. Zelinka and G. W. Weber, *Intelligent computing & optimization*, Springer **866** (2018).

[43] E. Del Rosario, B. Vitoriano and G. W. Weber, *OR for sustainable development*, (2020).

[44] E. Kropat, R. A. Tikidji-Hamburyan and G. W. Weber, *Operations research in neuroscience*, Annals of Operations Research **258**(1) (2017) 1–4.

[45] R. A. Tikidji-Hamburyan, E. Kropat and G. W. Weber, *OOperations Research in Neuroscience II*, Annals of Operations Research **289**(1) (2020) 1–152.

[46] E. Kropat, M. Türkay and G. W. Weber, *Introduction to the Special Issue on Fuzzy Analytics and Stochastic Methods in Neurosciences*, IEEE Transactions on Fuzzy Systems **28**(1) (2020) 1–4.